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COMPUTATION OF TWO-BODY MATRIX ELEMENTS FROM THE ARGONNE v_{18} POTENTIAL *

by

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Abstract. We discuss the computation of two-body matrix elements from the Argonne v_{18} interaction. The matrix elements calculation is presented both in particle-particle and in particle-hole angular momentum coupling. The procedures developed here can be applied to the case of other NN potentials, provided that they have a similar operator format.

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0. Introduction

The Argonne v_{18} potential [1] is an updated version of the nonrelativistic Argonne potential that fits both np data and pp data, as well as low-energy nn data scattering parameters and deuteron properties. The potential was directly fit to the Nijmegen NN scattering database, which contains 1787 pp and 2514 np data in the range 0-350 MeV, and has an excellent χ^2 per datum of 1.09 .

The strong interaction part of the potential is projected into an operator format with 18 terms: A charge-independent part that has 14 operator components (as in the older Argonne v_{14})

$$1, \quad \sigma_i \cdot \sigma_j, \quad S_{ij}, \quad L \cdot S, \quad L^2, \quad L^2 \sigma_i \cdot \sigma_j, \quad (L \cdot S)^2 \\ \tau_i \cdot \tau_j, \quad (\sigma_i \cdot \sigma_j) (\tau_i \cdot \tau_j), \quad S_{ij} (\tau_i \cdot \tau_j), \quad L \cdot S (\tau_i \cdot \tau_j), \quad L^2 (\tau_i \cdot \tau_j), \quad L^2 (\sigma_i \cdot \sigma_j) (\tau_i \cdot \tau_j), \quad (L \cdot S)^2 (\tau_i \cdot \tau_j)$$

And a charge-independence breaking part that has three charge-dependent operators

$$T_{ij}, \quad (\sigma_i \cdot \sigma_j) T_{ij}, \quad S_{ij} T_{ij}$$

where $T_{ij} = 3\tau_{zi}\tau_{zj} - \tau_i \cdot \tau_j$ is the isotensor operator, defined analogous to the S_{ij} operator; and one charge-asymmetric operator

$$\tau_{zi} + \tau_{zj}$$

The potential includes also a complete electromagnetic potential, containing Coulomb, Darwin-Foldy, vacuum polarization, and magnetic moment terms with finite-size effects.

Matrix elements of the NN interaction can be specified either in particle-particle (pp) coupling or in particle-hole (ph) coupling. Both of the matrix elements completely specify the interaction, and either set can be calculated from the other set. We first discuss the symmetries and the angular momentum couplings (AMC).

1. Symmetries of two-body matrix elements

Generally we use the definition: $k = j + 1/2$ and $\hat{j} = \sqrt{2j+1}$. The matrix element in m-representation can be written as

$$\langle m_a m_b | V | m_c m_d \rangle = V_{m_a m_b, m_c m_d} = \langle m_a m_c^{-1} | V | m_d m_b^{-1} \rangle = \langle m_a \bar{m}_c | V | m_d \bar{m}_b \rangle \quad 1.1$$

We use these notations interchangeably.

The angular momentum coupled matrix element is constructed as

$$\langle (j_a j_b)_L | V^L | (j_c j_d)_L \rangle = \sum_{m_a m_b m_c m_d} \langle j_a m_a j_b m_b | LM \rangle \langle j_c m_c j_d m_d | LM \rangle \langle m_a m_b | V | m_c m_d \rangle \quad 1.2$$

which is independent of M . It has the symmetry:

$$\langle (j_a j_b)_L | V^L | (j_c j_d)_L \rangle = (-)^{k_a + k_b + k_c + k_d} \langle (j_b j_a)_L | V^L | (j_d j_c)_L \rangle$$

The inverse relation is

$$\langle m_a m_b | V | m_c m_d \rangle = \sum_{LM} \langle j_a m_a j_b m_b | LM \rangle \langle j_c m_c j_d m_d | LM \rangle \langle (j_a j_b)_L | V^L | (j_c j_d)_L \rangle \quad 1.3$$

Instead of using this pp -coupling one can use ph -coupling to construct angular momentum coupled matrix elements. The ph -coupled matrix element is defined as

$$\langle (j_1 \bar{j}_2)_\lambda | V^\lambda | (j_3 \bar{j}_4)_\lambda \rangle = \sum_{m_1 m_2 m_3 m_4} (-)^{k_2 + m_2 + k_4 - m_4} \langle j_1 m_1 j_2 - m_2 | \lambda \mu \rangle \langle j_3 m_3 j_4 - m_4 | \lambda \mu \rangle \langle m_1 m_2^{-1} | V | m_3 m_4^{-1} \rangle \quad 1.4$$

These are again independent of μ .

The matrix elements in pp -coupling are related to the matrix elements in ph -coupling through [2]

$$\langle(j_1\bar{j}_2)_\lambda|V^\lambda|(j_3\bar{j}_4)_\lambda\rangle = \sum_L (-)^{k_2+k_3+L+1} (2L+1) \left\{ \begin{matrix} j_1 & j_2 & \lambda \\ j_3 & j_4 & L \end{matrix} \right\} \langle(j_1j_4)_L|V^L|(j_2j_3)_L\rangle \quad 1.5$$

This is obtained via general recoupling, using:

$$\begin{aligned} \sum_M \langle j_1m_1j_2m_2|LM\rangle \langle j_3m_3j_4m_4|LM\rangle &= \\ &= (-)^{j_2+m_1+j_4+m_3} \sum_{\lambda\mu} (2L+1) \left\{ \begin{matrix} j_1 & j_2 & L \\ j_3 & j_4 & \lambda \end{matrix} \right\} \langle j_1m_1j_4-m_4|\lambda\mu\rangle \langle j_3m_3j_2-m_2|\lambda\mu\rangle \\ &= (-)^{j_2-m_2+j_4+m_3+L} \sum_{\lambda\mu} (2L+1) \left\{ \begin{matrix} j_1 & j_2 & L \\ j_4 & j_3 & \lambda \end{matrix} \right\} \langle j_1m_1j_3-m_3|\lambda\mu\rangle \langle j_4m_4j_2-m_2|\lambda\mu\rangle \end{aligned}$$

The inverse relation for the matrix elements is

$$\langle(j_1j_4)_L|V^L|(j_2j_3)_L\rangle = (-)^{L+1} \sum_\lambda (-)^{k_2+k_3} (2\lambda+1) \left\{ \begin{matrix} j_1 & j_2 & \lambda \\ j_3 & j_4 & L \end{matrix} \right\} \langle(j_1\bar{j}_2)_\lambda|V^\lambda|(j_3\bar{j}_4)_\lambda\rangle \quad 1.6$$

Anti-symmetric matrix elements are formed in m-representation as

$$\langle m_a m_b | V^a | m_c m_d \rangle = \langle m_a m_b | V | m_c m_d \rangle - \langle m_a m_b | V | m_d m_c \rangle \quad 1.7$$

In pp angular momentum coupling this becomes

$$\langle(j_a j_b)_L|V^{aL}|(j_c j_d)_L\rangle = \langle(j_a j_b)_L|V^L|(j_c j_d)_L\rangle + (-)^{L+k_c+k_d} \langle(j_a j_b)_L|V^L|(j_d j_c)_L\rangle \quad 1.8$$

These anti-symmetric matrix elements have the symmetry

$$\langle(j_a j_b)_L|V^{aL}|(j_c j_d)_L\rangle = (-)^{L+k_c+k_d} \langle(j_a j_b)_L|V^{aL}|(j_d j_c)_L\rangle = (-)^{L+k_a+k_b} \langle(j_b j_a)_L|V^{aL}|(j_c j_d)_L\rangle \quad 1.9$$

In ph -coupling the anti-symmetrization can be obtained by transforming the anti-symmetric matrix elements in pp -coupling using (1.5). Thus we obtain

$$\begin{aligned} \langle(j_1\bar{j}_2)_\lambda|V^{a\lambda}|(j_3\bar{j}_4)_\lambda\rangle &= \langle(j_1\bar{j}_2)_\lambda|V^\lambda|(j_3\bar{j}_4)_\lambda\rangle + \\ &\sum_{\lambda'} (-)^{\lambda+\lambda'} (-)^{k_2+k_3} (2\lambda'+1) \left\{ \begin{matrix} j_1 & j_2 & \lambda \\ j_4 & j_3 & \lambda' \end{matrix} \right\} \langle(j_1\bar{j}_3)_{\lambda'}|V^{\lambda'}|(j_2\bar{j}_4)_{\lambda'}\rangle \end{aligned} \quad 1.10$$

The symmetry of the relations can be significantly improved by introducing the *Ring*-phase. For pp -coupling the *Ring*-phase is

$$\langle(j_a j_b)_L|V^{aLR}|(j_c j_d)_L\rangle = (-)^{k_a+k_c} \langle(j_a j_b)_L|V^{aL}|(j_c j_d)_L\rangle \quad 1.11$$

With this extra phase the matrix elements have the symmetry

$$\langle(j_a j_b)_L|V^{aLR}|(j_c j_d)_L\rangle = (-)^L \langle(j_a j_b)_L|V^{aLR}|(j_d j_c)_L\rangle = (-)^L \langle(j_b j_a)_L|V^{aLR}|(j_c j_d)_L\rangle \quad 1.12$$

and the anti-symmetric matrix elements can be calculated as

$$\langle(j_a j_b)_L|V^{aLR}|(j_c j_d)_L\rangle = \langle(j_a j_b)_L|V^{LR}|(j_c j_d)_L\rangle + (-)^L \langle(j_a j_b)_L|V^{LR}|(j_d j_c)_L\rangle \quad 1.13$$

For matrix elements in ph -coupling the *Ring*-phase is

$$\langle(j_1\bar{j}_2)_\lambda|V^{a\lambda R}|(j_3\bar{j}_4)_\lambda\rangle = (-)^{k_1+k_3} \langle(j_1\bar{j}_2)_\lambda|V^{a\lambda}|(j_3\bar{j}_4)_\lambda\rangle \quad 1.14$$

With this phase the anti-symmetric matrix elements in ph -coupling have the symmetry

$$\begin{aligned}\langle (j_1 \bar{j}_2)_\lambda | V^{a\lambda R} | (j_3 \bar{j}_4)_\lambda \rangle &= \langle (j_2 \bar{j}_1)_\lambda | V^{a\lambda R} | (j_4 \bar{j}_3)_\lambda \rangle \\ &= \sum_{\lambda'} (-)^{\lambda+\lambda'} (2\lambda' + 1) \left\{ \begin{matrix} j_1 & j_3 & \lambda' \\ j_4 & j_2 & \lambda \end{matrix} \right\} \langle (j_1 \bar{j}_3)_{\lambda'} | V^{a\lambda' R} | (j_2 \bar{j}_4)_{\lambda'} \rangle\end{aligned}\quad 1.14$$

Also, the anti-symmetric matrix elements in ph -coupling can be constructed as

$$\begin{aligned}\langle (j_1 \bar{j}_2)_\lambda | V^{a\lambda R} | (j_3 \bar{j}_4)_\lambda \rangle &= \langle (j_1 \bar{j}_2)_\lambda | V^{\lambda R} | (j_3 \bar{j}_4)_\lambda \rangle + \\ &\quad \sum_{\lambda'} (-)^{\lambda+\lambda'} (2\lambda' + 1) \left\{ \begin{matrix} j_1 & j_2 & \lambda \\ j_4 & j_3 & \lambda' \end{matrix} \right\} \langle (j_1 \bar{j}_3)_{\lambda'} | V^{\lambda' R} | (j_2 \bar{j}_4)_{\lambda'} \rangle\end{aligned}\quad 1.15$$

and the relation between pp -coupling and ph -coupling is simplified to

$$\langle (j_1 j_4)_L | V^{aLR} | (j_2 j_3)_L \rangle = (-)^{L+1} \sum_{\lambda} (2\lambda + 1) \left\{ \begin{matrix} j_1 & j_2 & \lambda \\ j_3 & j_4 & L \end{matrix} \right\} \langle (j_1 \bar{j}_2)_\lambda | V^{a\lambda R} | (j_3 \bar{j}_4)_\lambda \rangle\quad 1.16$$

or its reverse relation

$$\langle (j_1 \bar{j}_2)_\lambda | V^{a\lambda R} | (j_3 \bar{j}_4)_\lambda \rangle = \sum_L (-)^{L+1} (2L + 1) \left\{ \begin{matrix} j_1 & j_2 & \lambda \\ j_3 & j_4 & L \end{matrix} \right\} \langle (j_1 j_4)_L | V^{aLR} | (j_2 j_3)_L \rangle\quad 1.17$$

In all the nuclear structure calculations we will use anti-symmetric matrix elements that include the *Ring*-phase. For simplicity we will drop the superscripts aR throughout.

2. General considerations and relationships

We assume the interaction can be written as a scalar product of tensors of rank (k). Then the ph - ph matrix element including the *Ring*-phase of $(-)^{(k_1+k_3)}$ can be computed as

$$\begin{aligned}\langle (1, \bar{2})_\lambda | V^R | (3, \bar{4})_\lambda \rangle &= (-)^{(k_1+k_3)} \langle (1, \bar{2})_\lambda | V | (3, \bar{4})_\lambda \rangle = \\ &= (-)^{(k_1+k_3)} \langle (1, \bar{2})_\lambda | [U^{(k)}(1) \odot V^{(k)}(2)] | (3, \bar{4})_\lambda \rangle = \\ &= \delta_{k,\lambda} (-)^{k_1} \frac{1}{\lambda} < 1 \| U^{(\lambda)} \| 2 > \quad (-)^{k_4} \frac{1}{\lambda} < 4 \| V^{(\lambda)} \| 3 >\end{aligned}\quad 2.1$$

Thus it is necessary to bring the various interactions into this form.

We will use the Fourier transform to separate the variables \vec{r}_1 and \vec{r}_2 . For this, the general relation can be worked out :

$$\begin{aligned}V(r_{12}) Y_{J,M}(\hat{r}_{12}) &= \frac{2}{\pi} \int q^2 dq \tilde{V}^J(q) \sum_{\ell_1, \ell_2} \frac{\hat{\ell}_1 \hat{\ell}_2}{\hat{J}} \frac{1}{\sqrt{4\pi}} \langle \ell_1 0 \ell_2 0 | J 0 \rangle \\ &\quad \times j_{\ell_1}(qr_1) j_{\ell_2}(qr_2) (i)^{(\ell_1 - \ell_2 - J)} 4\pi [Y_{\ell_1}(\hat{r}_1) \otimes Y_{\ell_2}(\hat{r}_2)]^{(J,M)}\end{aligned}\quad 2.2$$

where the form factor of the interaction is

$$\tilde{V}^J(q) = \int V(r_{12}) j_J(qr_{12}) r_{12}^2 dr_{12}\quad 2.3$$

The various necessary operators are defined as

$$\begin{aligned}\vec{r}_{12} &= \vec{r}_1 - \vec{r}_2 \\ \vec{S} &= \frac{1}{2}(\vec{\sigma}_1 + \vec{\sigma}_2) \\ \vec{L} &= \frac{1}{(2i)} \vec{r}_{12} \times (\vec{\nabla}_1 - \vec{\nabla}_2)\end{aligned}\quad 2.4$$

Using ([3], eq. 5.1.4)

$$\hat{r}_{12}^{(1)} = \sqrt{\frac{4\pi}{3}} Y^{(1)}(\hat{r}_{12}) \quad 2.5$$

and ([3] eq. 5.2.4)

$$a^{(k)} \odot b^{(k)} = (-)^k \hat{k} [a^{(k)} \otimes b^{(k)}]^{(0)} \quad 2.6$$

Using ([3] eq. 5.1.8) we write

$$\begin{aligned} L^{(1)} &= \frac{1}{\sqrt{2}} r_{12} [\hat{r}_{12}^{(1)} \otimes (\vec{\nabla}_1 - \vec{\nabla}_2)^{(1)}]^{(1)} \\ &= r_{12} \sqrt{\frac{2\pi}{3}} [Y^{(1)}(\hat{r}_{12}) \otimes (\vec{\nabla}_1 - \vec{\nabla}_2)^{(1)}]^{(1)} \end{aligned} \quad 2.7$$

For terms containing L^2 we need to evaluate the commutator for the tensor components with $\vec{\nabla} = \vec{\nabla}_1 - \vec{\nabla}_2$ which takes the form

$$[\vec{r}_s^{(1)}, \vec{\nabla}_{-t}^{(1)}] = 2\delta_{s,t}$$

With this we obtain

$$\begin{aligned} \left[[\vec{r} \otimes \vec{\nabla}]^{(1)} \otimes [\vec{r} \otimes \vec{\nabla}]^{(1)} \right]^{(j)} &= 3 \sum_{J_r, J_p} \hat{J}_r \hat{J}_p \left\{ \begin{matrix} 1 & 1 & 1 \\ J_r & J_p & j \end{matrix} \right\} \left[[\vec{r} \otimes \vec{r}]^{(J_r)} \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(J_p)} \right]^{(j)} \\ &\quad + 6(-)^{(j+1)} \left\{ \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & j \end{matrix} \right\} [\vec{r} \otimes \vec{\nabla}]^{(j)} \end{aligned} \quad 2.8$$

Due to vanishing $\vec{a} \times \vec{a}$, each of j, J_r, J_p can take up only the values of 0 or 2. For this operator the following relation is useful

$$[\vec{r} \otimes \vec{r}]^{(J)} = r^2 \frac{4\pi}{3} [Y^{(1)}(\hat{r}) \otimes Y^{(1)}(\hat{r})]^{(J)} = r^2 \sqrt{\frac{4\pi}{2J+1}} < 1010 | J0 > Y^{(J)}(\hat{r}) \quad 2.9$$

Thus we can write

$$\begin{aligned} \left[[\vec{r} \otimes \vec{\nabla}]^{(1)} \otimes [\vec{r} \otimes \vec{\nabla}]^{(1)} \right]^{(j)} &= 3r^2 \sum_{J_r, J_p} \hat{J}_p \left\{ \begin{matrix} 1 & 1 & 1 \\ J_r & J_p & j \end{matrix} \right\} \langle 1 \ 0 \ 1 \ 0 | J_r \ 0 \rangle \sqrt{4\pi} [Y^{(J_r)}(\hat{r}) \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(J_p)}]^{(j)} \\ &\quad - 6r \left\{ \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & j \end{matrix} \right\} \sqrt{\frac{4\pi}{3}} [Y^{(1)}(\hat{r}) \otimes \vec{\nabla}]^{(j)} \end{aligned} \quad 2.10$$

3. Central interaction

Matrix elements for the interaction term $V^c(r_{12})$. This term includes the Coulomb interaction. The matrix element contributes only for natural parity matrix elements, i.e. $\lambda + l_1 + l_2 = \text{even}$. Including the “Ring”-phase of $(-)^{k_1+k_3}$ we get

$$\langle (1\bar{2})_\lambda | V^{c,R} | (3\bar{4})_\lambda \rangle = \frac{2}{\pi} \int q^2 dq F^c(q) \left(\frac{\sqrt{4\pi}}{\hat{\lambda}} (-)^{k_1} \langle 1 \| Y_\lambda \| 2 \rangle \frac{\sqrt{4\pi}}{\hat{\lambda}} (-)^{k_4} \langle 4 \| Y_\lambda \| 3 \rangle \right) \quad 3.1$$

where the form factor for the central interaction is

$$F^c(q) = \int V^c(r_{12}) j_0(qr_{12}) r_{12}^2 dr_{12} \quad 3.2$$

For the evaluation of the matrix element we need the following reduced matrix elements in which we include the radial matrix elements involving the Bessel functions: These also contain the *Ring*-phase. They are evaluated for terms that do not contain σ using ([3] eq. 7.1.7)

$$\frac{\sqrt{4\pi}}{\hat{k}} (-)^{k_1} \langle (\ell_1 s_1) j_1 \| G^{(k)} \| (\ell_2 s_2) j_2 \rangle = (-)^{(\ell_1+k_1+k_2)} \frac{\hat{j}_1 \hat{j}_2}{\hat{k}} \begin{Bmatrix} \ell_1 & \ell_2 & k \\ j_2 & j_1 & 1/2 \end{Bmatrix} \sqrt{4\pi} \langle \ell_1 \| G^{(k)} \| \ell_2 \rangle \quad 3.3$$

We abbreviate this as

$$f(k, 1, 2) \sqrt{4\pi} \langle \ell_1 \| G^{(k)} \| \ell_2 \rangle \quad 3.4$$

At this point we need the following reduced matrix elements ([3] eq. 5.4.5):

$$\sqrt{4\pi} \langle \ell_1 \| Y^\ell \| \ell_2 \rangle = (-)^{\ell_1} \hat{\ell}_1 \hat{\ell}_2 \langle \ell_1 0 \ell_2 0 | \ell 0 \rangle \langle R_1(r) | j_\lambda(qr) | R_2(r) \rangle \quad 3.5$$

We abbreviate this as

$$y(\ell, 1, 2) \langle R_1(r) | j_\lambda(qr) | R_2(r) \rangle \quad 3.6$$

With these definitions we write the reduced matrix elements as

$$\frac{\sqrt{4\pi}}{\hat{\lambda}} (-)^{k_1} \langle 1 \| Y^\lambda \| 2 \rangle = f(\lambda, 1, 2) y(\lambda, 1, 2) \langle R_1(r) | j_\lambda(qr) | R_2(r) \rangle \quad 3.7$$

4. Term (LL)

We write the interaction using eq.(2.8) with $j = 0$ as

$$\begin{aligned}
V &= V^{L2}(r_{12})[L \odot L] = -\sqrt{3}V^{L2}(r_{12})[L^{(1)} \otimes L^{(1)}]^{(0)} \\
&= -\frac{\sqrt{3}}{2}V^{L2}(r_{12})\left[\vec{r}_{12} \otimes \vec{\nabla}_{12}\right]^{(1)} \otimes \left[\vec{r}_{12} \otimes \vec{\nabla}_{12}\right]^{(1)}\bigg]^{(0)} \\
&= -\frac{\sqrt{3}}{2}r_{12}^2 V^{L2}(r_{12}) \sum_{J_r, J_p} \langle 1010 | J_r 0 \rangle \sqrt{4\pi} 3\hat{J}_p \begin{Bmatrix} 1 & 1 & 1 \\ J_r & J_p & 0 \end{Bmatrix} \left[Y^{(J_r)}(\hat{r}_{12}) \otimes [\vec{\nabla}_{12} \otimes \vec{\nabla}_{12}]^{(J_p)} \right]^{(0)} \\
&\quad + \frac{6}{2}\sqrt{4\pi} \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{Bmatrix} r_{12} V^{L2}(r_{12}) [Y^{(1)}(\hat{r}_{12}) \otimes \vec{\nabla}_{12}]^{(0)} \\
&= \frac{3}{2}\sqrt{4\pi} r_{12}^2 V^{L2}(r_{12}) \sum_J \langle 1010 | J 0 \rangle \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & J \end{Bmatrix} \left[Y^{(J)}(\hat{r}_{12}) \otimes [\vec{\nabla}_{12} \otimes \vec{\nabla}_{12}]^{(J)} \right]^{(0)} \\
&\quad - \sqrt{4\pi} r_{12} V^{L2}(r_{12}) [Y^{(1)}(\hat{r}_{12}) \otimes \vec{\nabla}_{12}]^{(0)}
\end{aligned} \tag{4.1}$$

As the $9j$ -coefficient requires $J_r = J_p$ we have set $J_r = J_p = J$. J can assume the values of 0 or 2. We employ the Fourier transform where we define the form factors of this interaction as

$$\tilde{V}^{L2,J}(q) = \int r_{12}^4 V^{L2}(r_{12}) j_J(qr_{12}) dr_{12} \tag{4.2}$$

and similarly

$$\bar{V}^{L2,1}(q) = \int r_{12}^3 V^{L2}(r_{12}) j_1(qr_{12}) dr_{12} \tag{4.3}$$

Then, using eq.(2.2) with eq.(4.1) we obtain

$$\begin{aligned}
V &= \sum_J \frac{3}{2} \langle 1010 | J 0 \rangle \frac{1}{\hat{J}} \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & J \end{Bmatrix} \sum_{\ell_1, \ell_2} (-)^{(\ell_1 + \ell_2 + J)/2} \hat{\ell}_1 \hat{\ell}_2 < \ell_1 0 \ell_2 0 | J 0 > \frac{2}{\pi} \int q^2 dq \tilde{V}^{L2J}(q) j_{\ell_1}(qr_1) j_{\ell_2}(qr_2) \\
&\quad \times (-)^{\ell_2} 4\pi \left[[Y^{(\ell_1)}(\hat{r}_1) \otimes Y^{(\ell_2)}(\hat{r}_2)]^{(J)} \otimes [\vec{\nabla}_{12} \otimes \vec{\nabla}_{12}]^{(J)} \right]^{(0)} \\
&+ \sum_{\ell_1, \ell_2} (-)^{(\ell_1 + \ell_2 + 1)/2} \frac{\hat{\ell}_1 \hat{\ell}_2}{\sqrt{3}} < \ell_1 0 \ell_2 0 | 1 0 > \frac{2}{\pi} \int q^2 dq \bar{V}^{L2}(q) j_{\ell_1}(qr_1) j_{\ell_2}(qr_2) \\
&\quad \times (-)^{\ell_2} 4\pi \left[[Y^{(\ell_1)}(\hat{r}_1) \otimes Y^{(\ell_2)}(\hat{r}_2)]^{(1)} \otimes \vec{\nabla}_{12} \right]^{(0)}
\end{aligned} \tag{4.4}$$

Substituting $\vec{\nabla}_{12} = \vec{\nabla}_1 - \vec{\nabla}_2$ results in five terms which need to be recoupled in order to be of the form $F^{(k)}(1) \odot G^{(k)}(2)$. These are

$$\begin{aligned}
Z_1 &= (-)^{\ell_2} \left[[Y^{(\ell_1)}(1) \otimes Y^{(\ell_2)}(2)]^{(J)} \otimes [\vec{\nabla}_1 \otimes \vec{\nabla}_1]^{(J)} \right]^{(0)} \\
&= \frac{1}{\hat{\ell}_2} \left[[Y^{(\ell_1)}(1) \otimes [\vec{\nabla}_1 \otimes \vec{\nabla}_1]^{(J)}]^{(\ell_2)} \odot Y^{(\ell_2)}(2) \right] \\
Z_2 &= (-2)(-)^{\ell_2} \left[[Y^{(\ell_1)}(1) \otimes Y^{(\ell_2)}(2)]^{(J)} \otimes [\vec{\nabla}_1 \otimes \vec{\nabla}_2]^{(J)} \right]^{(0)} \\
&= 2 \sum_k \hat{J} \begin{Bmatrix} \ell_1 & \ell_2 & J \\ 1 & 1 & k \end{Bmatrix} \left[[Y^{(\ell_1)}(1) \otimes \vec{\nabla}_1]^{(k)} \odot [Y^{(\ell_2)}(2) \otimes \vec{\nabla}_2]^{(k)} \right]
\end{aligned}$$

$$\begin{aligned}
Z_3 &= (-)^{\ell_2} \left[[Y^{(\ell_1)}(1) \otimes Y^{(\ell_2)}(2)]^{(J)} \otimes [\vec{\nabla}_2 \otimes \vec{\nabla}_2]^{(J)} \right]^{(0)} \\
&= \frac{1}{\hat{\ell}_1} \left[Y^{(\ell_1)}(1) \odot [Y^{(\ell_2)}(2) \otimes [\vec{\nabla}_2 \otimes \vec{\nabla}_2]^{(J)}]^{(\ell_1)} \right] \\
Z_4 &= (-)^{\ell_2} \left[[Y^{(\ell_1)}(1) \otimes Y^{(\ell_2)}(2)]^{(1)} \otimes \vec{\nabla}_1 \right]^{(0)} \\
&= \frac{1}{\hat{\ell}_2} \left[[Y^{(\ell_1)}(1) \otimes \vec{\nabla}_1]^{(\ell_2)} \odot Y^{(\ell_2)}(2) \right] \\
Z_5 &= -(-)^{\ell_2} \left[[Y^{(\ell_1)}(1) \otimes Y^{(\ell_2)}(2)]^{(1)} \otimes \vec{\nabla}_2 \right]^{(0)} \\
&= \frac{1}{\hat{\ell}_1} \left[Y^{(\ell_1)}(1) \odot [Y^{(\ell_2)}(2) \otimes \vec{\nabla}_2]^{(\ell_1)} \right]
\end{aligned} \tag{4.5}$$

For the evaluation of this matrix element we need the additional reduced matrix elements: The matrix element involving the operator ∇ :

$$\begin{aligned}
\sqrt{4\pi} < \ell_1 \| (Y^\ell \nabla)^{(\kappa)} \| \ell_2 > &= (-)^{\ell_2 + \kappa} \hat{\ell}_1 \hat{\ell}_2 \\
&\left\{ \sqrt{(2\ell_2 + 3)(\ell_2 + 1)} \begin{Bmatrix} \ell & 1 & \kappa \\ \ell_2 & \ell_1 & \ell_2 + 1 \end{Bmatrix} \begin{pmatrix} \ell_1 & \ell & \ell_2 + 1 \\ 0 & 0 & 0 \end{pmatrix} \langle R_1(r) | j_\ell(qr) \left(\frac{d}{dr} - \frac{\ell_2}{r} \right) | R_2(r) \rangle \right. \\
&\quad \left. - \sqrt{(2\ell_2 - 1)\ell_2} \begin{Bmatrix} \ell & 1 & \kappa \\ \ell_2 & \ell_1 & \ell_2 - 1 \end{Bmatrix} \begin{pmatrix} \ell_1 & \ell & \ell_2 - 1 \\ 0 & 0 & 0 \end{pmatrix} \langle R_1(r) | j_\ell(qr) \left(\frac{d}{dr} + \frac{\ell_2 + 1}{r} \right) | R_2(r) \rangle \right\}
\end{aligned} \tag{4.6}$$

which we abbreviate as

$$\sum_{i=1,2} del(i, \ell, \kappa, 1, 2) \langle R_1(r) | j_\ell(qr) \ op(i) | R_2(r) \rangle \tag{4.7}$$

where $op(1) = \frac{d}{dr}$ and $op(2) = \frac{1}{r}$.

And the matrix element involving the operator ∇^2 :

$$\begin{aligned}
\sqrt{4\pi} \langle \ell_1 \| [Y^{(\ell)} \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(J)}]^{(\lambda)} \| \ell_2 \rangle &= (-)^{(\ell_2 + \lambda + J)} \hat{\lambda} \hat{J} \hat{\ell}_1 \hat{\ell} \\
&\left[\begin{Bmatrix} \ell & J & \lambda \\ \ell_2 & \ell_1 & \ell_2 + 2 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & J \\ \ell_2 & \ell_2 + 2 & \ell_2 + 1 \end{Bmatrix} \begin{pmatrix} \ell_1 & \ell & \ell_2 + 2 \\ 0 & 0 & 0 \end{pmatrix} \right. \\
&\quad \times (\ell_2 + 2) \sqrt{\ell_2 + 1} \sqrt{\ell_2 + 2} < R_1(r) | j_\ell(qr) \left(\frac{d}{dr} - \frac{\ell_2 + 1}{r} \right) \left(\frac{d}{dr} - \frac{\ell_2}{r} \right) | R_2(r) > \\
&- \begin{Bmatrix} \ell & J & \lambda \\ \ell_2 & \ell_1 & \ell_2 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & J \\ \ell_2 & \ell_2 & \ell_2 + 1 \end{Bmatrix} \begin{pmatrix} \ell_1 & \ell & \ell_2 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad \times \hat{\ell}_2 (\ell_2 + 1) < R_1(r) | j_\ell(qr) \left(\frac{d}{dr} + \frac{\ell_2 + 2}{r} \right) \left(\frac{d}{dr} - \frac{\ell_2}{r} \right) | R_2(r) > \\
&- \begin{Bmatrix} \ell & J & \lambda \\ \ell_2 & \ell_1 & \ell_2 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & J \\ \ell_2 & \ell_2 & \ell_2 - 1 \end{Bmatrix} \begin{pmatrix} \ell_1 & \ell & \ell_2 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad \times \hat{\ell}_2 \ell_2 < R_1(r) | j_\ell(qr) \left(\frac{d}{dr} - \frac{\ell_2 - 1}{r} \right) \left(\frac{d}{dr} + \frac{\ell_2 + 1}{r} \right) | R_2(r) > \\
&+ \begin{Bmatrix} \ell & J & \lambda \\ \ell_2 & \ell_1 & \ell_2 - 2 \end{Bmatrix} \begin{Bmatrix} 1 & 1 & J \\ \ell_2 & \ell_2 - 2 & \ell_2 - 1 \end{Bmatrix} \begin{pmatrix} \ell_1 & \ell & \ell_2 - 2 \\ 0 & 0 & 0 \end{pmatrix} \\
&\quad \times (\ell_2 - 2) \sqrt{\ell_2 - 1} \sqrt{\ell_2} < R_1(r) | j_\ell(qr) \left(\frac{d}{dr} + \frac{\ell_2}{r} \right) \left(\frac{d}{dr} + \frac{\ell_2 + 1}{r} \right) | R_2(r) >
\end{aligned} \tag{4.8}$$

Here we give the reduced matrix elements separately for $J = 0$ and for $J = 2$ since they combine with different form factors. For $J = 0$ the selection rules require $\ell = \lambda$. Thus for $J = 0$ we can abbreviate (4.8) as

$$\sum_{i=1,3} ddel_0(i, \lambda, 1, 2) < R_1(r) | j_\ell(qr) op'(i) | R_2(r) > \tag{4.9}$$

whereas for $J = 2$ we abbreviate this as

$$\sum_{i=1,3} ddel_2(i, \ell, \lambda, 1, 2) < R_1(r) | j_\ell(qr) op'(i) | R_2(r) > \quad 4.10$$

with $op'(1) = \frac{d^2}{dr^2}$, $op'(2) = \frac{1}{r} \frac{d}{dr}$, and $op'(3) = \frac{1}{r^2}$.

Furthermore, we note that

$$\frac{3}{2} \langle 1010 | 00 \rangle \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{Bmatrix} = \frac{1}{\sqrt{12}} \quad 4.11$$

and

$$\frac{3}{2} \langle 1010 | 20 \rangle \begin{Bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \end{Bmatrix} \frac{1}{\sqrt{5}} = \sqrt{\frac{1}{120}} \quad 4.12$$

The matrix element of the $J = 0$ piece can be written as

$$\begin{aligned} < (1\bar{2})_\lambda | V_{(J=0)}^{R,L2} | (3\bar{4})_\lambda > = (-)^{k_1+k_4} \frac{2}{\pi} \int q^2 dq \tilde{V}^{L20}(q) \\ & \frac{1}{\sqrt{12}} \left(\frac{\sqrt{4\pi}}{\hat{\lambda}} < 1 \| (Y^\lambda \otimes [\nabla \otimes \nabla]^{(0)})^{(\lambda)} \| 2 > \frac{\sqrt{4\pi}}{\hat{\lambda}} < 4 \| Y^\lambda \| 3 > \right. \\ & \left. + \frac{\sqrt{4\pi}}{\hat{\lambda}} < 1 \| Y^\lambda \| 2 > \frac{\sqrt{4\pi}}{\hat{\lambda}} < 4 \| (Y^\lambda \otimes [\nabla \otimes \nabla]^{(0)})^{(\lambda)} \| 3 > \right) \end{aligned} \quad 4.13$$

This matrix element contributes only in natural parity cases.

$$\begin{aligned} < (1\bar{2})_\lambda | V_{Z=2, (J=0)}^{R,L2} | (3\bar{4})_\lambda > = \frac{4\pi}{2\lambda+1} (-)^{k_1+k_4} \frac{2}{\pi} \int q^2 dq \tilde{V}^{L20}(q) \\ \sum_{\ell=\lambda-1, \dots, \lambda+1} (-)^{\ell+\lambda+1} \frac{1}{3} < 1 \| (Y^\ell \nabla)^\lambda \| 2 > < 4 \| (Y^\ell \nabla)^\lambda \| 3 > \end{aligned} \quad 4.14$$

For the term with $J = 1$, linear in r , the contributions are from Z_4 and Z_5 :

$$\begin{aligned} < (1\bar{2})_\lambda | V_{(J=1)}^{R,L2} | (3\bar{4})_\lambda > = \\ \frac{1}{\sqrt{3}} \frac{4\pi}{2\lambda+1} (-)^{(k_1+k_4)} \sum_{\ell=\lambda-1, \lambda+1} (-)^{(\ell+\lambda+1)/2} \hat{\ell} < \ell 0 \lambda 0 | 10 > \frac{2}{\pi} \int q^2 dq \tilde{V}^{L2,1}(q) \\ \left(< 1 \| (Y^\ell \nabla)^\lambda \| 2 > < 4 \| Y^\lambda \| 3 > + < 1 \| Y^\lambda \| 2 > < 4 \| (Y^\ell \nabla)^\lambda \| 3 > \right) \end{aligned} \quad 4.15$$

This term contributes only in natural parity cases. We have to consider $\ell = \lambda - 1$ and $\ell = \lambda + 1$.

Finally, the contributions from $J = 2$ can be written

$$\begin{aligned} < (1\bar{2})_\lambda | V_{(J=2)Z=1}^{R,L2} | (3\bar{4})_\lambda > = -\frac{1}{\sqrt{120}} \frac{4\pi}{2\lambda+1} (-)^{(k_1+k_4)} \frac{2}{\pi} \int q^2 dq \tilde{V}^{L2,2}(q) \\ \sum_{\ell=\lambda-2, \lambda, \lambda+2} (-)^{(\ell+\lambda)/2} \hat{\ell} < \ell 0 \lambda 0 | 20 > \left(< 1 \| [Y^\ell (\nabla \otimes \nabla)^{(2)}]^{(\lambda)} \| 2 > < 4 \| Y^\lambda \| 3 > \right. \\ \left. + < 1 \| Y^\lambda \| 2 > < 4 \| [Y^\ell (\nabla \otimes \nabla)^{(2)}]^{(\lambda)} \| 3 > \right) \end{aligned} \quad 4.16$$

Again, this term contributes only in natural parity cases with $\ell = \lambda - 2$, $\ell = \lambda$, and $\ell = \lambda + 2$.

The remaining contribution is

$$\begin{aligned} < (1\bar{2})_\lambda | V_{(J=2)Z=2}^{R,L2} | (3\bar{4})_\lambda > = \\ -\frac{4\pi}{2\lambda+1} (-)^{(k_1+k_4)} \frac{2}{\pi} \int q^2 dq \tilde{V}^{L2,2}(q) \\ \sum_{\ell_a=\lambda\pm 1} \sum_{\ell_b=\lambda\pm 1} d_{lsq}(\ell_a, \ell_b, \lambda) < 1 \| (Y^{\ell_a} \nabla)^\lambda \| 2 > < 4 \| (Y^{\ell_b} \nabla)^\lambda \| 3 > \end{aligned} \quad 4.17$$

with

$$d_{lsq}(\ell_a, \ell_b, \lambda) = \sqrt{\frac{1}{6}} (-)^{(\ell_a + \ell_b)/2} \hat{\ell}_a \hat{\ell}_b < \ell_a 0 \ell_b 0 | 20 > \left\{ \begin{matrix} 1 & 1 & 2 \\ \ell_a & \ell_b & \lambda \end{matrix} \right\} \quad 4.18$$

This term contributes in both, natural and unnatural parity cases. We have to consider the three cases with $\ell_a = \ell_b = \lambda - 1$, λ , and $\lambda + 1$ as well as the two cases: $\ell_a = \lambda - 1, \ell_b = \lambda + 1$ and $\ell_a = \lambda + 1, \ell_b = \lambda - 1$.

With our abbreviations the reduced matrix elements can be written as

$$(-)^{k_1} \frac{\sqrt{4\pi}}{\hat{\lambda}} \langle 1 || (Y^\ell \vec{\nabla})^\lambda || 2 \rangle = \sum_{i=1,2} f(\lambda, 1, 2) del(i, \ell, \lambda, 1, 2) \langle R_1(r) | j_\ell(qr) \text{ op}(i) | R_2(r) \rangle \quad 4.19$$

$$\frac{\sqrt{4\pi}}{\hat{\lambda}} (-)^{k_1} < 1 || [Y^\lambda (\nabla \otimes \nabla)^{(0)}]^{(\lambda)} || 2 > = \sum_{i=1,3} f(\lambda, 1, 2) ddel_0(i, \lambda, 1, 2) \langle R_1(r) | j_\ell(qr) \text{ op}'(i) | R_2(r) \rangle \quad 4.20$$

and

$$\frac{\sqrt{4\pi}}{\hat{\lambda}} (-)^{k_1} < 1 || [Y^\ell (\nabla \otimes \nabla)^{(2)}]^{(\lambda)} || 2 > = \sum_{i=1,3} f(\lambda, 1, 2) ddel_2(i, \ell, \lambda, 1, 2) \langle R_1(r) | j_\ell(qr) \text{ op}'(i) | R_2(r) \rangle \quad 4.21$$

5. Terms $\sigma\sigma$ and $(\mathbf{LL})\sigma\sigma$

Such terms can be obtained from the previous terms such as the central interaction or the L^2 -interaction by adding the $\sigma_1\sigma_2$ term. This results in

$$V = V^{L2\sigma}(r_{12}) [L \odot L] [\vec{\sigma}_1 \odot \vec{\sigma}_2] \quad 5.1$$

We can obtain the separated interaction immediately from the separated form for the L^2 term via

$$[F^{(J)}(1) \odot G^{(J)}(2)] [\vec{\sigma}_1 \odot \vec{\sigma}_2] = \sum_k (-)^{(k+J+1)} \left[[F^{(J)}(1) \otimes \vec{\sigma}_1]^{(k)} \odot [G^{(J)}(2) \otimes \vec{\sigma}_2]^{(k)} \right] \quad 5.2$$

Thus, we can immediately take the results of the previous sections and write the matrix element for the σ -interaction as

$$\begin{aligned} < (1\bar{2})_\lambda | V_{(J=0)}^{R,\sigma} | (3\bar{4})_\lambda > = \frac{4\pi}{2\lambda+1} (-)^{k_1+k_4} \frac{2}{\pi} \int q^2 dq \tilde{V}^\sigma(q) \\ & \sum_{\ell=\lambda-1, \dots, \lambda+1} (-)^{(\ell+\lambda+1)} < 1 || (Y^\ell \sigma)^\lambda || 2 > < 4 || (Y^\ell \sigma)^\lambda || 3 > \end{aligned} \quad 5.3$$

Similarly we write the contributions to the matrix elements of the $L^2\sigma$ -interaction for $J = 0$ as

$$\begin{aligned} < (1\bar{2})_\lambda | V_{(J=0)}^{R,L2\sigma} | (3\bar{4})_\lambda > = \frac{1}{\sqrt{12}} \frac{4\pi}{2\lambda+1} (-)^{k_1+k_4} \frac{2}{\pi} \int q^2 dq \tilde{V}^{L2\sigma}(q) \sum_{\ell=\lambda-1, \dots, \lambda+1} (-)^{(\ell+\lambda+1)} \\ & \left(< 1 || \left[(Y^\ell \otimes [\nabla \otimes \nabla]^{(0)})^{(\ell)} \sigma \right]^\lambda || 2 > < 4 || (Y^\ell \sigma)^\lambda || 3 > \right. \\ & \left. + < 1 || (Y^\ell \sigma)^\lambda || 2 > < 4 || \left[(Y^\ell \otimes [\nabla \otimes \nabla]^{(0)})^\ell \sigma \right]^{(\lambda)} || 3 > \right) \end{aligned} \quad 5.4$$

We include three cases: $\ell = \lambda - 1$, $\ell = \lambda$, and $\ell = \lambda + 1$.

$$\begin{aligned} < (1\bar{2})_\lambda | V_{Z=2(J=0)}^{R,L2\sigma} | (3\bar{4})_\lambda > = \frac{1}{3} \frac{4\pi}{2\lambda+1} (-)^{k_1+k_4} \frac{2}{\pi} \int q^2 dq \tilde{V}^{L2\sigma}(q) \\ & \sum_{\kappa} \sum_{\ell=\kappa-1, \kappa+1} (-)^{(\ell+\lambda)} < 1 || \left[(Y^\ell \nabla)^\kappa \sigma \right]^{(\lambda)} || 2 > < 4 || \left[(Y^\ell \nabla)^\kappa \sigma \right]^{(\lambda)} || 3 > \end{aligned} \quad 5.5$$

In this term ℓ can take the values of $\lambda - 2$, $\lambda - 1$, λ , $\lambda + 1$, and $\lambda + 2$.

For the term with $J = 1$, linear in r the contributions are from Z_4 and Z_5 :

$$\begin{aligned} & \langle (1\bar{2})_\lambda | V_{(J=1)}^{R,L2\sigma} | (3\bar{4})_\lambda \rangle = \\ & \sum_{\kappa} \sum_{\ell=\kappa-1, \kappa+1} (-)^{(\ell+\kappa+1)/2} (-)^{(\kappa+\lambda+1)} \frac{1}{\sqrt{3}} \frac{4\pi}{2\lambda+1} (-)^{(k_1+k_4)} \hat{\ell} \langle \ell 0 \kappa 0 | 10 \rangle \frac{2}{\pi} \int q^2 dq \tilde{V}^{L2\sigma,1}(q) \\ & \left(\langle 1 \| [(Y^\ell \nabla)^{(\kappa)} \sigma]^{(\lambda)} \| 2 \rangle \langle 4 \| (Y^\kappa \sigma)^{(\lambda)} \| 3 \rangle + \langle 1 \| (Y^\kappa \sigma)^{(\lambda)} \| 2 \rangle \langle 4 \| [(Y^\ell \nabla)^{(\kappa)} \sigma]^{(\lambda)} \| 3 \rangle \right) \end{aligned} \quad 5.6$$

Here κ can be $\lambda - 1$, λ , and $\lambda + 1$. For each case ℓ can be $\kappa + 1$ and $\kappa - 1$.

Finally, the contributions from $J = 2$ can be written

$$\begin{aligned} & \langle (1\bar{2})_\lambda | V_{(J=2)Z=1}^{R,L2\sigma} | (3\bar{4})_\lambda \rangle = \sum_{\kappa} (-)^{(\kappa+\lambda)} \frac{1}{\sqrt{120}} \frac{4\pi}{2\lambda+1} (-)^{(k_1+k_4)} \frac{2}{\pi} \int q^2 dq \tilde{V}^{L2\sigma,2}(q) \\ & \sum_{\ell=\kappa-2, \kappa, \kappa+2} (-)^{(\ell+\kappa)/2} \hat{\ell} \langle \ell 0 \kappa 0 | 20 \rangle \left(\langle 1 \| [Y^\ell (\nabla \otimes \nabla)^{(2)}]^{(\kappa)} \sigma]^{(\lambda)} \| 2 \rangle \langle 4 \| (Y^\kappa \sigma)^{(\lambda)} \| 3 \rangle \right. \\ & \left. + \langle 1 \| (Y^\kappa \sigma)^{(\lambda)} \| 2 \rangle \langle 4 \| [Y^\ell (\nabla \otimes \nabla)^{(2)}]^{(\kappa)} \sigma]^{(\lambda)} \| 3 \rangle \right) \end{aligned} \quad 5.7$$

Again, κ can take the values $\lambda - 1$, λ , and $\lambda + 1$. For each case ℓ can be κ , $\kappa - 2$, and $\kappa + 2$.

$$\begin{aligned} & \langle (1\bar{2})_\lambda | V_{(J=2)Z=2}^{R,L2\sigma} | (3\bar{4})_\lambda \rangle = \sum_{\kappa} (-)^{(\kappa+\lambda)} \frac{1}{\sqrt{120}} \frac{4\pi}{2\lambda+1} (-)^{(k_1+k_4)} \frac{2}{\pi} \int q^2 dq \tilde{V}^{L2\sigma,2}(q) \\ & + \sum_{\ell_a=\kappa\pm 1} \sum_{\ell_b=\kappa\pm 1} d\ell(\ell_a, \ell_b, \kappa) \langle 1 \| [Y^{\ell_a} \nabla]^{(\kappa)} \sigma]^{(\lambda)} \| 2 \rangle \langle 4 \| [Y^{\ell_b} \nabla]^{(\kappa)} \sigma]^{(\lambda)} \| 3 \rangle \end{aligned} \quad 5.8$$

For the reduced one-body matrix elements containing σ we use ([3] eq. 7.1.5)

$$\frac{\sqrt{4\pi}}{\hat{\lambda}} (-)^{k_1} \langle (\ell_1 s_1) j_1 \| (G^{(\ell)} \sigma)^{(\lambda)} \| (\ell_2 s_2) j_2 \rangle = (-)^{k_1} \hat{j}_1 \hat{j}_2 \sqrt{6} \begin{Bmatrix} \ell_1 & \ell_2 & \ell \\ j_1 & j_2 & \lambda \end{Bmatrix} \sqrt{4\pi} \langle \ell_1 \| G^{(\ell)} \| \ell_2 \rangle \quad 5.9$$

we abbreviate this as

$$\sum_{\ell} g(\ell, k, 1, 2) \sqrt{4\pi} \langle \ell_1 \| G^{(\ell)} \| \ell_2 \rangle \quad 5.10$$

With our abbreviations the reduced matrix elements can be written as

$$(-)^{k_1} \frac{\sqrt{4\pi}}{\hat{\lambda}} \langle 1 \| (Y^\ell \vec{\sigma})^\lambda \| 2 \rangle = g(\ell, \lambda, 1, 2) y(\ell, 1, 2) \langle R_1(r) | j_\ell(qr) | R_2(r) \rangle \quad 5.11$$

$$\frac{\sqrt{4\pi}}{\hat{\lambda}} (-)^{k_1} \langle 1 \| [Y^\ell \otimes \vec{\nabla}]^\kappa \otimes \vec{\sigma}^\lambda \| 2 \rangle = \sum_{i=1,2} g(\kappa, \lambda, 1, 2) del(i, \ell, \kappa, 1, 2) \langle R_1(r) | j_\ell(qr) \ op(i) | R_2(r) \rangle \quad 5.12$$

$$\frac{\sqrt{4\pi}}{\hat{\lambda}} (-)^{k_1} \langle 1 \| [Y^\kappa (\nabla \otimes \nabla)^{(0)}]^{(\kappa)} \sigma]^{(\lambda)} \| 2 \rangle = \sum_{i=1,3} g(\kappa, \lambda, 1, 2) ddel_0(i, \kappa, 1, 2) \langle R_1(r) | j_\ell(qr) \ op'(i) | R_2(r) \rangle \quad 5.13$$

and

$$\frac{\sqrt{4\pi}}{\hat{\lambda}} (-)^{k_1} \langle 1 \| [Y^\ell (\nabla \otimes \nabla)^{(2)}]^{(\kappa)} \sigma]^{(\lambda)} \| 2 \rangle = \sum_{i=1,3} g(\kappa, \lambda, 1, 2) ddel_2(i, \ell, \kappa, 1, 2) \langle R_1(r) | j_\ell(qr) \ op'(i) | R_2(r) \rangle \quad 5.14$$

6. Spin-orbit (LS)

$$\begin{aligned} V &= V^{LS}(r_{12})[L^{(1)} \odot S^{(1)}] = -\sqrt{3}V^{LS}(r_{12})[L^{(1)} \otimes S^{(1)}]^{(0)} \\ &= -r_{12}V^{LS}(r_{12})\sqrt{2\pi}\left[[Y^{(1)}(\hat{r}_{12}) \otimes (\vec{\nabla}_1 - \vec{\nabla}_2)^{(1)}]^{(1)} \otimes S^{(1)}\right]^{(0)} \end{aligned} \quad 6.1$$

We introduce the form-factor of the LS-interaction:

$$\tilde{V}^{LS}(q) = \int r_{12}^3 V^{LS}(r_{12}) j_1(qr_{12}) dr_{12} \quad 6.2$$

and write using eq.(2.2) with $J = 1$

$$\begin{aligned} V &= -\sqrt{2\pi} \sum_{\ell_1, \ell_2} \frac{2}{\pi} \int q^2 dq \tilde{V}^{LS}(q) j_{\ell_1}(qr_1) j_{\ell_2}(qr_2) \\ &\quad \times \hat{\ell}_1 \hat{\ell}_2 \frac{1}{\sqrt{3 \cdot 4\pi}} \langle \ell_1 0 \ell_2 0 | 10 \rangle (i)^{\ell_1} (-i)^{\ell_2+1} \\ &\quad \times 4\pi \left[[Y^{(\ell_1)}(\hat{r}_1) \otimes Y^{(\ell_2)}(\hat{r}_2)]^{(1)} \otimes (\vec{\nabla}_1 - \vec{\nabla}_2)^{(1)} \right]^{(1)} \otimes S^{(1)} \Big]^{(0)} \end{aligned} \quad 6.3$$

Or, combining factors

$$\begin{aligned} V &= \sum_{\ell_1, \ell_2} (-)^{(\ell_1 - \ell_2 + 1)/2} \sqrt{\frac{1}{24}} \frac{2}{\pi} \int q^2 dq \tilde{V}^{LS}(q) j_{\ell_1}(qr_1) j_{\ell_2}(qr_2) \\ &\quad \times \hat{\ell}_1 \hat{\ell}_2 \langle \ell_1 0 \ell_2 0 | 10 \rangle \\ &\quad \times 4\pi \left[[Y^{(\ell_1)}(\hat{r}_1) \otimes Y^{(\ell_2)}(\hat{r}_2)]^{(1)} \otimes (\vec{\nabla}_1 - \vec{\nabla}_2)^{(1)} \right]^{(1)} \otimes (\sigma_1 + \sigma_2)^{(1)} \Big]^{(0)} \end{aligned} \quad 6.4$$

Inserting S from above, results in four terms as represented in the last line of the previous equation. They must be recoupled and brought into the form $F^{(k)}(1) \odot G^{(k)}(2)$. These terms are:

$$\begin{aligned} Z_1 &= \left[[Y^{\ell_1}(1) \otimes Y^{\ell_2}(2)]^{(1)} \otimes \vec{\nabla}_1^{(1)} \right]^{(1)} \otimes \sigma_1^{(1)} \Big]^{(0)} \\ &= -\sum_k \sqrt{3} \hat{k} \begin{Bmatrix} \ell_2 & \ell_1 & 1 \\ 1 & 1 & k \end{Bmatrix} \left[Y^{\ell_2}(2) \otimes [Y^{\ell_1}(1) \otimes \vec{\nabla}_1]^{(k)} \otimes \sigma_1 \right]^{(\ell_2)} \Big]^{(0)} \\ &= (-)^{(\ell_2+1)} \sum_k \sqrt{3} \frac{\hat{k}}{\hat{\ell}_2} \begin{Bmatrix} \ell_2 & \ell_1 & 1 \\ 1 & 1 & k \end{Bmatrix} \left[Y^{\ell_2}(2) \odot [Y^{\ell_1}(1) \otimes \vec{\nabla}_1]^{(k)} \otimes \sigma_1 \right]^{(\ell_2)} \Big] \\ Z_2 &= (-) \left[[Y^{\ell_1}(1) \otimes Y^{\ell_2}(2)]^{(1)} \otimes \vec{\nabla}_2^{(1)} \right]^{(1)} \otimes \sigma_1^{(1)} \Big]^{(0)} \\ &= (-)^{\ell_2} \sqrt{3} \sum_k \begin{Bmatrix} \ell_2 & \ell_1 & 1 \\ 1 & 1 & k \end{Bmatrix} \left[[Y^{\ell_1}(1) \otimes \sigma_1]^{(k)} \odot [Y^{\ell_2}(2) \otimes \nabla_2]^{(k)} \right] \\ Z_3 &= \left[[Y^{\ell_1}(1) \otimes Y^{\ell_2}(2)]^{(1)} \otimes \vec{\nabla}_1^{(1)} \right]^{(1)} \otimes \sigma_2^{(1)} \Big]^{(0)} \\ &= (-)^{\ell_2} \sqrt{3} \sum_k \begin{Bmatrix} \ell_2 & \ell_1 & 1 \\ 1 & 1 & k \end{Bmatrix} \left[[Y^{\ell_2}(2) \otimes \sigma_2]^{(k)} \odot [Y^{\ell_1}(1) \otimes \nabla_1]^{(k)} \right] \\ Z_4 &= (-) \left[[Y^{\ell_1}(1) \otimes Y^{\ell_2}(2)]^{(1)} \otimes \vec{\nabla}_2^{(1)} \right]^{(1)} \otimes \sigma_2^{(1)} \Big]^{(0)} \\ &= (-)^{(\ell_2+1)} \sum_k \sqrt{3} \frac{\hat{k}}{\hat{\ell}_1} \begin{Bmatrix} \ell_2 & \ell_1 & 1 \\ 1 & 1 & k \end{Bmatrix} \left[Y^{\ell_1}(1) \odot [Y^{\ell_2}(2) \otimes \vec{\nabla}_2]^{(k)} \otimes \sigma_2 \right]^{(\ell_1)} \Big] \end{aligned} \quad 6.5$$

by introducing

$$d_{so}(\ell_1, \ell_2, \lambda) = \frac{-1}{\sqrt{8}} \hat{\ell}_1 \hat{\lambda} \langle \ell_1 \ 0 \ \ell_2 \ 0 | 1 \ 0 \rangle (-)^{(\ell_1 + \ell_2 + 1)/2} \begin{Bmatrix} 1 & 1 & 1 \\ \ell_1 & \ell_2 & \lambda \end{Bmatrix} \quad 6.6$$

we write the matrix element as

$$\begin{aligned} \langle (1, \bar{2})_\lambda | V^{R, LS} | (3, \bar{4})_\lambda \rangle &= \frac{4\pi}{2\lambda + 1} (-)^{(k_1 + k_4)} \sum_{\ell = \lambda \pm 1} \frac{2}{\pi} \int q^2 dq \tilde{V}^{LS}(q) \\ &\quad \left(\sum_k d_{so}(\ell, \lambda, k) \langle 1 \| [Y^\ell \otimes \vec{\nabla}]^{(k)} \otimes \sigma \| 2 \rangle \langle 4 \| Y^\lambda \| 3 \rangle \right. \\ &\quad + \sum_k d_{so}(\ell, \lambda, k) \langle 4 \| [Y^\ell \otimes \vec{\nabla}]^{(k)} \otimes \sigma \| 3 \rangle \langle 1 \| Y^\lambda \| 2 \rangle \\ &\quad - d_{so}(\ell, \lambda, \lambda) \langle 1 \| (Y^\ell \sigma)^{(\lambda)} \| 2 \rangle \langle 4 \| (Y^\lambda \nabla)^{(\lambda)} \| 3 \rangle \\ &\quad - d_{so}(\ell, \lambda, \lambda) \langle 4 \| (Y^\ell \sigma)^{(\lambda)} \| 3 \rangle \langle 1 \| (Y^\lambda \nabla)^{(\lambda)} \| 2 \rangle \\ &\quad - d_{so}(\ell, \lambda, \lambda) \langle 1 \| (Y^\lambda \sigma)^{(\lambda)} \| 2 \rangle \langle 4 \| (Y^\ell \nabla)^{(\lambda)} \| 3 \rangle \\ &\quad \left. - d_{so}(\ell, \lambda, \lambda) \langle 4 \| (Y^\lambda \sigma)^{(\lambda)} \| 3 \rangle \langle 1 \| (Y^\ell \nabla)^{(\lambda)} \| 2 \rangle \right) \end{aligned} \quad 6.7$$

All the reduced one body matrix elements have been defined before.

7. Tensor interaction

We write the tensor operator S_{12} as

$$S_{12} = 3(\vec{\sigma}_1 \hat{r}_{12})(\vec{\sigma}_2 \hat{r}_{12}) - \vec{\sigma}_1 \vec{\sigma}_2 \quad 7.1$$

Recoupling this, S_{12} can be written as coupled tensor operator

$$S_{12} = 3 \sum_J \hat{J} \left[[\sigma_1 \otimes \sigma_2]^{(J)} \otimes [\hat{r}_{12} \otimes \hat{r}_{12}]^{(J)} \right]^{(0)} + \sqrt{3} [\sigma_1 \otimes \sigma_2]^{(0)} \quad 7.2$$

with only $J = 0$ and $J = 2$ contributing. In this sum the last term is cancelled against the $J = 0$ term in the sum. The remaining term is

$$S_{12} = \sqrt{4\pi} \ 3 \langle 1010 | 20 \rangle \left[Y^2(\hat{r}_{12}) \otimes [\sigma_1 \otimes \sigma_2]^{(2)} \right]^{(0)} \quad 7.3$$

Separating \hat{r}_{12} and evaluating the Clebsch-Gordan coefficient ($3 \langle 1010 | 20 \rangle = \sqrt{6}$) finally gives

$$\begin{aligned} \langle (1, \bar{2})_\lambda | V^{R, T} | (3, \bar{4})_\lambda \rangle &= \frac{4\pi}{2\lambda + 1} (-)^{(k_1 + k_4)} \sum_{\ell_a} \sum_{\ell_b} \frac{2}{\pi} \int q^2 dq \tilde{V}^T(q) \\ &\quad \sqrt{6} \hat{\ell}_a \hat{\ell}_b \langle \ell_a \ 0 \ \ell_b \ 0 | 20 \rangle (-)^{(\ell_a + \ell_b)/2} \begin{Bmatrix} \ell_a & \ell_b & 2 \\ 1 & 1 & \lambda \end{Bmatrix} \langle 1 \| (Y^{\ell_a} \sigma)^{(\lambda)} \| 2 \rangle \langle 4 \| (Y^{\ell_b} \sigma)^{(\lambda)} \| 3 \rangle \end{aligned} \quad 7.4$$

where the form factor is

$$\tilde{V}^T(q) = \int V^T(r_{12}) j_2(qr_{12}) r_{12}^2 dr_{12} \quad 7.5$$

8. Term $(\mathbf{LS})^2$

This term can be brought into a similar form as the some of the interactions already discussed. We write

$$V = V^{LS^2}(r_{12}) [\vec{L} \odot \vec{S}] [\vec{L} \odot \vec{S}] = V^{LS^2}(r_{12}) \sum_j \hat{j} \left[[L^{(1)} \otimes L^{(1)}]^{(j)} \otimes [S^{(1)} \otimes S^{(1)}]^{(j)} \right]^{(0)} \quad 8.1$$

Here j can assume the values 0, 1 and 2. While the terms with $j = 0, 1$ can immediately be deduced from the previous cases, the term with $j = 2$ needs to be evaluated new.

For $j = 0$ we write

$$[\vec{L} \otimes \vec{L}]^{(0)} \otimes [\vec{S} \otimes \vec{S}]^{(0)} = \frac{1}{3} [\vec{L} \odot \vec{L}] [\vec{S} \odot \vec{S}] = \frac{1}{6} [\vec{L} \odot \vec{L}] (3 + \vec{\sigma}_1 \cdot \vec{\sigma}_2) \quad 8.2$$

Thus this part can be added immediately to the previously evaluated interactions by using $V^{L2,eff} = V^{L2} + \frac{1}{2} V^{LS^2}$ and $V^{L2\sigma,eff} = V^{L2\sigma} + \frac{1}{6} V^{LS^2}$.

For $j = 1$ we write

$$\sqrt{3} \left[[L^{(1)} \otimes L^{(1)}]^{(1)} \otimes [S^{(1)} \otimes S^{(1)}]^{(1)} \right]^{(0)} = -\frac{1}{2} [\vec{L} \odot \vec{S}] \quad 8.3$$

This contribution is incorporated by substituting $V^{LS,eff} = V^{LS} - \frac{1}{2} V^{LS^2}$.

Here it remains to evaluate only the matrix elements of

$$V = V^{LS^2}(r_{12}) \sqrt{5} \left[[L^{(1)} \otimes L^{(1)}]^{(2)} \otimes [S^{(1)} \otimes S^{(1)}]^{(2)} \right]^{(0)} \quad 8.4$$

Using eq.(2.4) we write

$$[S^{(1)} \otimes S^{(1)}]^{(2)} = \frac{1}{4} \left([\vec{\sigma}_1 \otimes \vec{\sigma}_1]^{(2)} + 2[\vec{\sigma}_1 \otimes \vec{\sigma}_2]^{(2)} + [\vec{\sigma}_2 \otimes \vec{\sigma}_2]^{(2)} \right)$$

Noting that one-body matrix elements vanish for the operator $[\sigma_1 \otimes \sigma_1]^{(2)}$ we write the remaining terms as

$$V = V^{LS^2}(r_{12}) \frac{\sqrt{5}}{2} \left[[L^{(1)} \otimes L^{(1)}]^{(2)} \otimes [\sigma_1^{(1)} \otimes \sigma_2^{(1)}]^{(2)} \right]^{(0)} \quad 8.5$$

We assume that similar to eq.(4.4) the operator $[L \otimes L]^{(2)}$ can be written as a tensor product of variables \vec{r}_1 and \vec{r}_2 as $[U^{(\kappa_1)}(1) \otimes W^{(\kappa_2)}(2)]^{(2)}$. Thus we can write

$$\begin{aligned} \left[[\vec{L} \otimes \vec{L}]^{(2)} \otimes [\vec{\sigma}_1 \otimes \vec{\sigma}_2]^{(2)} \right]^{(0)} &= \left[[U^{(\kappa_1)}(1) \otimes W^{(\kappa_2)}(2)] \otimes [\vec{\sigma}_1 \otimes \vec{\sigma}_2]^{(2)} \right]^{(0)} \\ &= \sqrt{5} \sum_k (-)^{(k_2+1)} \left\{ \begin{matrix} 1 & 1 & 2 \\ \kappa_1 & \kappa_2 & k \end{matrix} \right\} \left[[U^{(\kappa_1)}(1) \otimes \vec{\sigma}_1]^{(k)} \odot [W^{(\kappa_2)}(2) \otimes \vec{\sigma}_2]^{(k)} \right]^{(0)} \end{aligned} \quad 8.6$$

We now have to factorize the operator $[L \otimes L]^{(2)}$ using eq.(1.10) with $j = 2$. Again, as they combine with different form factors, we list the three cases separately namely: (a) $J_r = 0, J_p = 2$, (b) $J_r = 2, J_p = 0$, and (c) $J_r = 2, j_p = 2$. Numerically evaluating the 9j-symbol and CG-coefficient we find

$$\begin{aligned} \frac{\sqrt{5}}{2} [\vec{L} \otimes \vec{L}]^{(2)} &= \sqrt{4\pi} \left(\frac{\sqrt{5}}{12} r^2 [Y^{(0)}(\hat{r}) \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(2)}]^{(2)} - \frac{\sqrt{2}}{24} r^2 [Y^{(2)}(\hat{r}) \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(0)}]^{(2)} \right. \\ &\quad \left. + \frac{\sqrt{14}}{12} r^2 [Y^{(2)}(\hat{r}) \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(2)}]^{(2)} - \frac{1}{4} \sqrt{\frac{5}{3}} r [Y^{(1)}(\hat{r}) \otimes \vec{\nabla}]^{(2)} \right) \end{aligned} \quad 5.7$$

In the following we list all the eleven terms that contribute to $[L \otimes L]^{(2)}$ in their recoupled forms:

$$\begin{aligned}
Z_1 &= \left[[Y^{\ell_1}(1) \otimes Y^{\ell_2}(2)]^{(2)} \otimes [\nabla_1 \otimes \nabla_1]^{(0)} \right]^{(2)} \\
&= \left[[Y^{\ell_1} \otimes [\nabla_1 \otimes \nabla_1]^{(0)}]^{(\ell_1)} \otimes Y^{\ell_2}(2) \right]^{(2)} \\
Z_2 &= -2 \left[[Y^{\ell_1}(1) \otimes Y^{\ell_2}(2)]^{(2)} \otimes [\nabla_1 \otimes \nabla_2]^{(0)} \right]^{(2)} \\
&= \frac{2}{\sqrt{3}} \sum_{L_1, L_2} (-)^{(L_2 + \ell_1)} \hat{L}_1 \hat{L}_2 \begin{Bmatrix} L_1 & L_2 & 2 \\ \ell_2 & \ell_1 & 1 \end{Bmatrix} \left[[Y^{\ell_1}(1) \nabla_1]^{(L_1)} \otimes [Y^{\ell_2}(2) \nabla_2]^{(L_2)} \right]^{(2)} \\
Z_3 &= \left[Y^{\ell_1}(1) \otimes [Y^{\ell_2}(2) \otimes [\nabla_2 \otimes \nabla_2]^{(0)}]^{(\ell_2)} \right]^{(2)} \\
Z_4 &= \left[[Y^{\ell_1}(1) \otimes Y^{\ell_2}(2)]^{(0)} \otimes [\nabla_1 \otimes \nabla_1]^{(2)} \right]^{(2)} \\
&= \sum_k \delta_{\ell_1, \ell_2} \frac{\hat{k}}{\hat{\ell}_1 \sqrt{5}} \left[[Y^{\ell_1}(1) \otimes [\nabla_1 \otimes \nabla_1]^{(2)}]^{(k)} \otimes Y^{\ell_2}(2) \right]^{(2)} \\
Z_5 &= -2 \left[[Y^{\ell_1}(1) \otimes Y^{\ell_2}(2)]^{(0)} \otimes [\nabla_1 \otimes \nabla_2]^{(2)} \right]^{(2)} \\
&= 2\delta_{\ell_1, \ell_2} \sum_{L_1, L_2} (-)^{(L_1 + \ell_1)} \frac{\hat{L}_1 \hat{L}_2}{\ell_1} \begin{Bmatrix} 1 & 1 & 2 \\ L_2 & L_1 & \ell_1 \end{Bmatrix} \left[[Y^{\ell_1}(1) \nabla_1]^{(L_1)} \otimes [Y^{\ell_2}(2) \nabla_2]^{(L_2)} \right]^{(2)} \\
Z_6 &= \sum_k (-)^{(k + \ell_2)} \delta_{\ell_1, \ell_2} \frac{\hat{k}}{\hat{\ell}_1 \sqrt{5}} \left[Y^{\ell_1}(1) \otimes [Y^{\ell_2}(2) \otimes [\nabla_2 \otimes \nabla_2]^{(2)}]^{(k)} \right]^{(2)} \\
Z_7 &= \left[[Y^{\ell_1}(1) \otimes Y^{\ell_2}(2)]^{(2)} \otimes [\nabla_1 \otimes \nabla_1]^{(2)} \right]^{(2)} \\
&= \sqrt{5} \sum_k (-)^{(\ell_1 + k)} \hat{k} \begin{Bmatrix} 2 & 2 & 2 \\ \ell_2 & \ell_1 & k \end{Bmatrix} \left[[Y^{\ell_1}(1) \otimes [\nabla_1 \otimes \nabla_1]^{(2)}]^{(k)} \otimes Y^{\ell_2}(2) \right]^{(2)} \\
Z_8 &= -2 \left[[Y^{\ell_1}(1) \otimes Y^{\ell_2}(2)]^{(2)} \otimes [\nabla_1 \otimes \nabla_2]^{(2)} \right]^{(2)} \\
&= -10 \sum_{L_1, L_2} \hat{L}_1 \hat{L}_2 \begin{Bmatrix} \ell_1 & \ell_2 & 2 \\ 1 & 1 & 2 \\ L_1 & L_2 & 2 \end{Bmatrix} \left[[Y^{\ell_1}(1) \nabla_1]^{(L_1)} \otimes [Y^{\ell_2}(2) \nabla_2]^{(L_2)} \right]^{(2)} \\
Z_9 &= \sqrt{5} \sum_k \hat{k} \begin{Bmatrix} 2 & 2 & 2 \\ \ell_2 & \ell_1 & k \end{Bmatrix} \left[Y^{\ell_1}(1) \otimes [Y^{\ell_2}(2) \otimes [\nabla_2 \otimes \nabla_2]^{(2)}]^{(k)} \right]^{(2)} \\
Z_{10} &= \left[[Y^{\ell_1} \otimes Y^{\ell_2}]^{(1)} \otimes \nabla_1 \right]^{(2)} \\
&= \sum_k (-)^{(\ell_2 + k)} \sqrt{3} \hat{k} \begin{Bmatrix} 1 & 2 & 1 \\ \ell_2 & \ell_1 & k \end{Bmatrix} \left[[Y^{\ell_1}(1) \nabla_1]^{(k)} \otimes Y^{\ell_2}(2) \right]^{(2)} \\
Z_{11} &= - \sum_k \sqrt{3} \hat{k} \begin{Bmatrix} 1 & 2 & 1 \\ \ell_1 & \ell_2 & k \end{Bmatrix} \left[Y^{\ell_1}(1) \otimes [Y^{\ell_2}(2) \nabla_2]^{(k)} \right]^{(2)}
\end{aligned} \tag{8.8}$$

Combining this with with eq.(8.7), we can write the matrix element contributions for the various form factors ($J = 0$, $J = 1$, and $J = 2$) as

$$\begin{aligned}
\langle (1\bar{2})_\lambda | V_{(J=0)}^{R, LS2} | (3\bar{4})_\lambda \rangle &= -\frac{\sqrt{5}}{24} \frac{4\pi}{2\lambda + 1} (-)^{(k_1 + k_4)} \frac{2}{\pi} \int q^2 dq \tilde{V}^{LS20}(q) \sum_{\ell, \kappa} \hat{\kappa} \begin{Bmatrix} 1 & 1 & 2 \\ \kappa & \ell & \lambda \end{Bmatrix} \\
&\quad \left(\langle 1 \| [(Y^\ell \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(2)})^{(\kappa)} \vec{\sigma}]^{(\lambda)} \| 2 \rangle \langle 4 \| (Y^\ell \vec{\sigma})^{(\lambda)} \| 3 \rangle \right. \\
&\quad \left. + \langle 1 \| (Y^\ell \vec{\sigma})^{(\lambda)} \| 2 \rangle \langle 4 \| [(Y^\ell \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(2)})^{(\kappa)} \vec{\sigma}]^{(\lambda)} \| 3 \rangle \right)
\end{aligned} \tag{8.9}$$

$$\begin{aligned}
\langle (1\bar{2})_\lambda | V_{(J=0)Z=2}^{R,LS2} | (3\bar{4})_\lambda \rangle &= -\frac{5}{12} \frac{4\pi}{2\lambda+1} (-)^{(k_1+k_4)} \sum_{\ell, \ell_a, \ell_b} (-)^{(\ell_a+\ell_b)} \hat{\ell}_a \hat{\ell}_b \left\{ \begin{matrix} 1 & 1 & 2 \\ \ell_a & \ell_b & \ell \end{matrix} \right\} \left\{ \begin{matrix} 1 & 1 & 2 \\ \ell_a & \ell_b & \lambda \end{matrix} \right\} \\
&\quad \frac{2}{\pi} \int q^2 dq \tilde{V}^{LS20}(q) \times \langle 1 \| [(Y^\ell \vec{\nabla})^{(\ell_a)} \vec{\sigma}]^{(\lambda)} \| 2 \rangle \langle 4 \| [(Y^\ell \vec{\nabla})^{(\ell_b)} \vec{\sigma}]^{(\lambda)} \| 3 \rangle
\end{aligned} \tag{8.10}$$

The contributions from terms Z_{10} , and Z_{11} , the $J = 1$ terms are

$$\begin{aligned}
\langle (1\bar{2})_\lambda | V_{(J=1)}^{R,LS2} | (3\bar{4})_\lambda \rangle &= \frac{5}{\sqrt{12}} \frac{4\pi}{2\lambda+1} (-)^{(k_1+k_4)} \sum_{\ell_a, \ell_b, \kappa} (-)^\kappa \hat{\kappa} \langle \ell_a 0 \ell_b 0 | 10 \rangle \left\{ \begin{matrix} 1 & 1 & 2 \\ \ell_b & \kappa & \ell_a \end{matrix} \right\} \left\{ \begin{matrix} 1 & 1 & 2 \\ \ell_b & \kappa & \lambda \end{matrix} \right\} \\
&\quad \frac{2}{\pi} \int q^2 dq \tilde{V}^{LS2,1}(q) \hat{\ell}_a \hat{\ell}_b (-)^{(\ell_a-\ell_b-1)/2} \left(\langle 1 \| [(Y^{\ell_a} \vec{\nabla})^{(\kappa)} \vec{\sigma}]^{(\lambda)} \| 2 \rangle \langle 4 \| (Y^{\ell_b} \vec{\sigma})^{(\lambda)} \| 3 \rangle \right. \\
&\quad \left. + \langle 1 \| (Y^{\ell_b} \vec{\sigma})^{(\lambda)} \| 2 \rangle \langle 4 \| [(Y^{\ell_a} \vec{\nabla})^{(\kappa)} \vec{\sigma}]^{(\lambda)} \| 3 \rangle \right)
\end{aligned} \tag{8.11}$$

Finally, the $J = 2$ terms are from Z_1 , Z_2 , Z_3 , Z_7 , Z_8 , and Z_9 .

$$\begin{aligned}
\langle (1\bar{2})_\lambda | V_{(J=2)}^{R,LS2} | (3\bar{4})_\lambda \rangle &= \frac{4\pi}{2\lambda+1} (-)^{(k_1+k_4)} \frac{2}{\pi} \int q^2 dq \tilde{V}^{LS2,2}(q) \sum_{\ell_1, \ell_2} \hat{\ell}_1 \hat{\ell}_2 \langle \ell_1 0 \ell_2 0 | 20 \rangle (-)^{(\ell_1+\ell_2)/2} \\
&\quad \left\{ \sum_{\kappa} \frac{\sqrt{70}}{24} \hat{\kappa} \left\{ \begin{matrix} 1 & 1 & 2 \\ \kappa & \ell_1 & \lambda \end{matrix} \right\} \left\{ \begin{matrix} 2 & 2 & 2 \\ \ell_1 & \ell_2 & \kappa \end{matrix} \right\} (-)^{(\kappa+\ell_1)} \right. \\
&\quad \left(\langle 1 \| [Y^{\ell_2} \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(2)}]^{(\kappa)} \vec{\sigma}]^{(\lambda)} \| 2 \rangle \langle 4 \| (Y^{\ell_1} \vec{\sigma})^{(\lambda)} \| 3 \rangle \right. \\
&\quad \left. + \langle 1 \| (Y^{\ell_1} \vec{\sigma})^{(\lambda)} \| 2 \rangle \langle 4 \| [Y^{\ell_2} \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(2)}]^{(\kappa)} \vec{\sigma}]^{(\lambda)} \| 3 \rangle \right) \\
&\quad - \frac{\sqrt{2}}{24} \left\{ \begin{matrix} 1 & 1 & 2 \\ \ell_1 & \ell_2 & \lambda \end{matrix} \right\} \left(\langle 1 \| [Y^{\ell_2} \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(0)}]^{(0)} \vec{\sigma}]^{(\lambda)} \| 2 \rangle \langle 4 \| (Y^{\ell_1} \vec{\sigma})^{(\lambda)} \| 3 \rangle \right. \\
&\quad \left. \left. + \langle 1 \| (Y^{\ell_1} \vec{\sigma})^{(\lambda)} \| 2 \rangle \langle 4 \| [Y^{\ell_2} \otimes [\vec{\nabla} \otimes \vec{\nabla}]^{(0)}]^{(0)} \vec{\sigma}]^{(\lambda)} \| 3 \rangle \right) \right\} \\
\langle (1\bar{2})_\lambda | V_{(J=2)}^{R,LS2} | (3\bar{4})_\lambda \rangle &= -\frac{4\pi}{2\lambda+1} (-)^{(k_1+k_4)} \frac{2}{\pi} \int q^2 dq \tilde{V}^{LS2,2}(q) \sum_{\ell_1, \ell_2} \hat{\ell}_1 \hat{\ell}_2 \langle \ell_1 0 \ell_2 0 | 20 \rangle (-)^{(\ell_1+\ell_2)/2} \\
&\quad \sum_{\ell_a, \ell_b} \hat{\ell}_a \hat{\ell}_b \left\{ \begin{matrix} 1 & 1 & 2 \\ \ell_a & \ell_b & \lambda \end{matrix} \right\} \left(\sqrt{\frac{2}{3}} \frac{1}{6} \left\{ \begin{matrix} \ell_2 & \ell_1 & 2 \\ \ell_a & \ell_b & \lambda \end{matrix} \right\} + \sqrt{14} \frac{5}{6} (-)^{(\ell_b+\ell_2)} \left\{ \begin{matrix} \ell_1 & \ell_2 & 2 \\ 1 & 1 & 2 \\ \ell_a & \ell_b & 2 \end{matrix} \right\} \right) \\
&\quad \langle 1 \| [(Y^{\ell_1} \vec{\nabla})^{(\ell_a)} \vec{\sigma}]^{(\lambda)} \| 2 \rangle \langle 4 \| [(Y^{\ell_2} \vec{\nabla})^{(\ell_b)} \vec{\sigma}]^{(\lambda)} \| 3 \rangle
\end{aligned} \tag{8.12}$$

8.13

9. The isospin component

The isospin dependence of the matrix elements is best worked out using the (pp) coupled matrix elements. In that case all we have to do is to work out the expectation value of the isospin operator. We do this here for the four isospin operators that appear in the Argonne v_{18} potential of Wiringa *et al* [1] and the four possible matrix elements ($\mathbf{1}$, $\vec{\tau}_1 \vec{\tau}_2$, $T_{12} = 3\tau_{z1}\tau_{z2} - \vec{\tau}_1 \vec{\tau}_2$, and $\tau_{z1} + \tau_{z2}$):

$$\begin{aligned}
\langle (pp) | \mathbf{1} | (pp) \rangle &= +1 & , & & \langle (pp) | \tau_{z1} + \tau_{z2} | (pp) \rangle &= +2 \\
\langle (pn) | \mathbf{1} | (pn) \rangle &= +1 & , & & \langle (pn) | \tau_{z1} + \tau_{z2} | (pn) \rangle &= 0 \\
\langle (pn) | \mathbf{1} | (np) \rangle &= 0 & , & & \langle (pn) | \tau_{z1} + \tau_{z2} | (np) \rangle &= 0 \\
\langle (nn) | \mathbf{1} | (nn) \rangle &= +1 & , & & \langle (nn) | \tau_{z1} + \tau_{z2} | (nn) \rangle &= -2 \\
\langle (pp) | \vec{\tau}_1 \vec{\tau}_2 | (pp) \rangle &= +1 & , & & \langle (pp) | T_{12} | (pp) \rangle &= +2 \\
\langle (pn) | \vec{\tau}_1 \vec{\tau}_2 | (pn) \rangle &= -1 & , & & \langle (pn) | T_{12} | (pn) \rangle &= -2 \\
\langle (pn) | \vec{\tau}_1 \vec{\tau}_2 | (np) \rangle &= +2 & , & & \langle (pn) | T_{12} | (np) \rangle &= -2 \\
\langle (nn) | \vec{\tau}_1 \vec{\tau}_2 | (nn) \rangle &= +1 & , & & \langle (nn) | T_{12} | (nn) \rangle &= +2
\end{aligned}$$

Thus we calculate the matrix elements
for $\langle (pp)|V|(pp) \rangle$

$$\langle (12)|V|(34) \rangle = \langle (12)|V + V^{\tau\tau} + 2V^{tT} + 2V^{\tau z} + V^{Coul}|(34) \rangle$$

for $\langle (pn)|V|(pn) \rangle$

$$\langle (12)|V|(34) \rangle = \langle (12)|V - V^{\tau\tau} - 2V^{tT}|(34) \rangle$$

for $\langle (pn)|V|(np) \rangle$

$$\langle (12)|V|(34) \rangle = \langle (12)|2V^{\tau\tau} - 2V^{tT}|(34) \rangle$$

and for $\langle (nn)|V|(nn) \rangle$

$$\langle (12)|V|(34) \rangle = \langle (12)|V + V^{\tau\tau} + 2V^{tT} - 2V^{\tau z}|(34) \rangle$$

10. CM-correction terms

From the center-of-mass (CM) corrections we have two additional terms that must be included in the two-body matrix elements of the interaction. These are:

$$-\frac{\vec{p}_1 \cdot \vec{p}_2}{mA} = +\frac{\hbar^2 \vec{\nabla}_1 \cdot \vec{\nabla}_2}{mA}$$

and

$$\frac{\vec{r}_1 \cdot \vec{r}_2}{mA}$$

These can be computed using eq.(2.1). These matrix elements contribute only for $\lambda = 1$ and can be computed with eq.(3.3) resulting in

$$-\frac{1}{mA} \langle 1\bar{2}|\vec{p}_1 \cdot \vec{p}_2|3\bar{4} \rangle = \frac{\hbar^2}{mA} (-)^{k_1} \frac{1}{\sqrt{3}} \langle 1\|\nabla\|2 \rangle (-)^{k_4} \frac{1}{\sqrt{3}} \langle 4\|\nabla\|3 \rangle$$

and

$$\frac{1}{mA} \langle 1\bar{2}|\vec{r}_1 \cdot \vec{r}_2|3\bar{4} \rangle = \frac{1}{mA} (-)^{k_1} \frac{1}{\sqrt{3}} \langle 1\|\vec{r}\|2 \rangle (-)^{k_4} \frac{1}{\sqrt{3}} \langle 4\|\vec{r}\|3 \rangle$$

Here

$$(-)^{k_1} \frac{1}{\sqrt{3}} \langle 1\|\vec{\nabla}\|2 \rangle = \frac{\hat{j}_1 \hat{j}_2}{3} \langle j_1 \frac{1}{2} j_2 \frac{-1}{2} | 10 \rangle \langle R_1(r) | \frac{d}{dr} + \frac{1}{2}(\ell_2 - \ell_1)(3\ell_2 - \ell_1 + 1) \frac{1}{r} | R_2(r) \rangle$$

and

$$(-)^{k_1} \frac{1}{\sqrt{3}} \langle 1\|\vec{r}\|2 \rangle = -\frac{\hat{j}_1 \hat{j}_2}{3} \langle j_1 \frac{1}{2} j_2 \frac{-1}{2} | 10 \rangle \langle R_1(r) | r | R_2(r) \rangle$$

11. The radial integrals

We assume that each wave function is expanded in harmonic oscillator functions corresponding to the oscillator length parameter ok .

$$R_{i,l}(r) = \sum_n A_n^i H_{n,l}(r) \quad 10.1$$

As such each of the radial wave functions can be written as a polynomial multiplied by a gaussian of argument $-(\sqrt{2}r/ok)^2$. We now introduce the variable $x = \sqrt{2}r/ok$. With this, each radial wave function can be written in terms of x as

$$R_{i,l}(x) = \bar{R}_{i,l}(x) e^{-x^2/4} \quad 10.2$$

where $\bar{R}_{i,l}$ is a ploynomial in x . Generally, we define the barred functions as those without the exponential. In terms of x the wave functions are normalized such that

$$\int R_{i,l}^2(x) x^2 dx = \left(\frac{\sqrt{2}}{ok} \right)^3 \quad 10.3$$

In order to carry out the Fourier transform we use the fact that the Fourier transforms of the harmonic oscillator functions are again harmonic oscillator functions in k -space. Thus we expand the product of two radial functions as

$$R_{i,l_i}(x) R_{j,l_j}(x) = \sum_n B_n^{i,j} H_{n,L}(x) \quad 10.4$$

Using Gaussian integration, the expansion coefficients can be written as

$$B_n^{i,j} = \sum_k \bar{H}_{n,L}(x_k) \bar{R}_{i,l_i}(x_k) \bar{R}_{j,l_j}(x_k) x_k^2 w_k \quad 10.5$$

with this, we write the integral

$$\int R_{i,l_i}(r) j_L(qr) R_{j,l_j}(r) r^2 dr = \sum_n B_n^{i,j} H_{n,L}(\bar{q}) \quad 10.6$$

where $\bar{q} = \sqrt{2}ok \ q$. The radial integrals can now be computed according to

$$\begin{aligned} \frac{2}{\pi} \int q^2 dq \ V(q) \langle R_{i,l_i} | j_L(qr) | R_{j,l_j} \rangle \langle R_{s,l_s} | j_K(qr) | R_{t,l_t} \rangle = \\ = \sum_n \sum_m B_n^{i,j} B_m^{s,t} \frac{2}{\pi} \int q^2 dq \ V(q) H_{n,L}(\bar{q}) H_{m,K}(\bar{q}) \\ =: \sum_n \sum_m B_n^{i,j} B_m^{s,t} I_{n,m}^{L,K} \end{aligned} \quad 10.7$$

Inserting the expression for the expansion coefficients B we find

$$\begin{aligned} \frac{2}{\pi} \int q^2 dq \ V(q) \langle R_{i,l_i} | j_L(qr) | R_{j,l_j} \rangle \langle R_{s,l_s} | j_K(qr) | R_{t,l_t} \rangle = \\ = \frac{2}{\pi} \sum_k \sum_q \bar{R}_{i,l_i}(x_k) \bar{R}_{j,l_j}(x_k) x_k^2 w_k \bar{R}_{s,l_s}(x_q) \bar{R}_{t,l_t}(x_q) x_q^2 w_q \sum_{m,n} I_{n,m}^{L,K} \bar{H}_{n,L}(x_k) \bar{H}_{m,K}(x_q) \\ =: \sum_k \sum_q \bar{R}_{i,l_i}(x_k) \bar{R}_{j,l_j}(x_k) \bar{R}_{s,l_s}(x_q) \bar{R}_{t,l_t}(x_q) S_{k,q}^{L,K} \end{aligned} \quad 10.8$$

With this notation all terms can be computed as

$$\begin{aligned} < (1\bar{2})_\lambda | V^{tot} | (3\bar{4})_\lambda > = \sum_{i,j} \bar{R}_1(x_i) \bar{R}_2(x_i) S_{i,j}^1 \bar{R}_3(x_j) \bar{R}_4(x_j) \\ & + \bar{R}_1(x_i) \bar{R}_2'(x_i) S_{i,j}^2 \bar{R}_3(x_j) \bar{R}_4(x_j) \\ & + \bar{R}_1(x_i) \bar{R}_2(x_i) S_{i,j}^3 \bar{R}_3'(x_j) \bar{R}_4(x_j) \\ & + \bar{R}_1(x_i) \bar{R}_2''(x_i) S_{i,j}^4 \bar{R}_3(x_j) \bar{R}_4(x_j) \\ & + \bar{R}_1(x_i) \bar{R}_2(x_i) S_{i,j}^5 \bar{R}_3''(x_j) \bar{R}_4(x_j) \\ & + \bar{R}_1(x_i) \bar{R}_2'(x_i) S_{i,j}^6 \bar{R}_3'(x_j) \bar{R}_4(x_j) \end{aligned} \quad 10.9$$

Correspondingly, the exchange contribution can be written as

$$\begin{aligned}
< (1\bar{2})_\lambda | V^{tot} | (3\bar{4})_\lambda > = \sum_{i,j} \bar{R}_1(x_i) \bar{R}_3(x_i) E_{i,j}^1 \bar{R}_2(x_j) \bar{R}_4(x_j) \\
& + \bar{R}_1(x_i) \bar{R}'_3(x_i) E_{i,j}^2 \bar{R}_2(x_j) \bar{R}_4(x_j) \\
& + \bar{R}_1(x_i) \bar{R}_3(x_i) E_{i,j}^3 \bar{R}'_2(x_j) \bar{R}_4(x_j) \\
& + \bar{R}_1(x_i) \bar{R}''_3(x_i) E_{i,j}^4 \bar{R}_2(x_j) \bar{R}_4(x_j) \\
& + \bar{R}_1(x_i) \bar{R}_3(x_i) E_{i,j}^5 \bar{R}''_2(x_j) \bar{R}_4(x_j) \\
& + \bar{R}_1(x_i) \bar{R}'_3(x_i) E_{i,j}^6 \bar{R}'_2(x_j) \bar{R}_4(x_j)
\end{aligned} \tag{10.10}$$

As the matrices S and E only depend on the angular momenta involved but not on the quantum numbers n_1, n_2, n_3 , and n_4 , the strategy is to compute the matrices first and then compute all the matrix elements with the same angular momentum combinations. This way a large number of matrix elements can be computed in which orbits differ only by the quantum number n .

Note. This strategy is essential in computing the G-matrix interaction in calculating the term:

$$\sum_K \sum_{p_3, p_4} Z_{p_3 p_4, h_1 h_2}^K < (p_3 p_4)_K | V | (p_1 p_2)_K >$$

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